Functional Analysis I

Department of Mathematics and Computer Applications

Chapter One ⇒ Basic Concepts

Chapter Two ⇒ Hilbert Spaces

Chapter Three ⇒ Orthogonal Complement and direct sums

Chapter Four ⇒ Continuous Linear Mappings.

BOOKS:

1. Introductory Functional Analysis with Applications, by Erwin Kerysing.

2. Introduction to Hilbert Spaces, by Sterling K. Berberian

3. Introduction to Functional Analysis with Applications, by A. H. Siddiqi, Khalil Ahmad and P. Manchanda.

Contents

1 Basic Concepts **2**

1.1 Metric Spaces 2

1.2 Basic Principles of Topology 6

1.3 Convergence in Metric Spaces 10

1.4 Vector Spaces 12

2 Hilbert Spaces **15**

2.1 Normed Spaces 15

2.2 Convergence in Normed Spaces 20

2.3 Inner Product Spaces or Pre-Hilbert Spaces 22

3 Orthogonal complements and direct sums **32**

3.1 Orthogonal Vectors 32

3.2 The Distance in pre-Hilbert space 40

3.3 Direct Sum Decompotions 44

Chapter 1

Basic Concepts

1.1 Metric Spaces

In this chapter we shall study more general "spaces" and "functions" defined on them. We arrive at a sufficiently general and flexible concept of a "space" as follows. We replace the set of real numbers underlying $\mathbb{R}$ by an abstract set X (set of elements whose nature is left unspecified) and introduce on X a "distance function" which has only a few of the most fundamental properties of the distance function on $\mathbb{R}$.

Definition 1.1.1. Let X be a non-empty set. Suppose that $d : X \times X \to \mathbb{R}^+$ is a function. Then (X, d) is a **metric space** if the following conditions are satisfies for all $x, y, x \in X$:
(M1) d is real valued, finite and non-negative.
(M2) $d(x, y) = 0$ if and only if $x = y$.
(M3) $d(x, y) = d(y, x)$ **(Symmetry)**.
(M4) $d(x, y) \leq d(x, z) + d(z, y)$ **(Triangle inequality)**.
The function d is called a **metric on** X or **distance function on** X.

The elements of X are called **points**. For fixed $x, y \in X$ the non-negative $d(x, y)$ is called the **distance from** x **to** y.

Example 1.1.2 (Real Line $\mathbb{R}$)**.**
Let $X = \mathbb{R}$ and $d(x, y) = |x - y|$ for all $x, y \in X$. Then (X, d) is a metric space.

Let $x, y, z \in \mathbb{R}$. Then:

- It is clear that d is real valued, finite and non-negative.
- If $x = y$ then $d(x, y) = |x - y| = 0$.

- $d(x,y) = |x-y| = |y-x| = d(y,x)$.
- $d(x,y) = |x-y| = |x-z+z-y| \leq |x-z| + |z-y| = d(x,z) + d(z,y)$.

Example 1.1.3 (Euclidean plane $\mathbb{R}^2$)**.**

The metric space $\mathbb{R}^2$, called the **Euclidean plane**, is obtained if we take the set of ordered pairs of real numbers, written $x = (x_1, x_2), y = (y_1, y_2)$, etc. Take d as follows

$$d(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}.$$

Then $(\mathbb{R}^2, d)$ is a metric space. d is called the **Euclidean metric**.

Another metric space is obtained if we choose that same set as before but another metric d_1 defined by

$$d_1(x,y) = |x_1 - y_1| + |x_2 - y_2|.$$

Let $x, y, z \in \mathbb{R}^2$. Then

- It is clear that d_1 is real valued, finite and non-negative.
- If $x = y$ then $(x_1, x_2) = (y_1, y_2)$. Thus $d_1(x,y) = 0$.
- $d_1(x,y) = |x_1 - y_1| + |x_2 - y_2| = |y_1 - x_1| + |y_2 - x_2| = d_1(y,x)$.
- $$\begin{aligned} d_1(x,y) &= |x_1 - z_1 + z_1 - y_1| + |x_2 - z_2 + z_2 - y_2| \\ &\leq |x_1 - z_1| + |z_1 - y_1| + |x_2 - z_2| + |z_2 - y_2| \\ &= d_1(x,z) + d_1(z,y) \end{aligned}$$

Example 1.1.4 (Euclidean plane $\mathbb{R}^n$)**.**

The previous example is a special case of n-dimensional Euclidean space $\mathbb{R}^n$. This space is obtained if we take the set of all ordered n-tuples of real numbers, written

$$x = (x_1, x_2, \cdots, x_n), \qquad y = (y_1, y_2, \cdots, y_n), etc.$$

$(\mathbb{R}^n, d)$, $(\mathbb{R}^n, d_1)$ and $(\mathbb{R}^n, d_\infty)$ are metric spaces where

$$d(x,y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2} = \sqrt{\sum_{k=1}^{n} (x_k - y_k)^2},$$

and

$$d_1(x,y) = |x_1 - y_1| + \cdots + |x_n - y_n| = \sum_{k=1}^{n} |x_k - y_k|$$

and

$$d_\infty(x,y) = \max_{i=1\in\mathbb{N}} |x_i - y_i|.$$

- - - -

Example 1.1.6 (Discrete metric space)**.** Let X be any non-empty set. The discrete metric on X defined by

$$d(x,y) := \begin{cases} 0, & \text{when } x = y, \\ 1, & \text{when } x \neq y. \end{cases}$$

for all $x, y \in X$. Then (X, d) is a metric space.

Let $x, y, z \in X$. Then:

- It is clear that d is real valued, finite and non-negative.
- If $x = y$ then $d(x, y) = 0$.
- $d(x, y) = 1 = d(y, x)$
- $d(x, y) = 1 < d(x, z) + d(z, y) = 2$.

Remark 1.1.7.

(1) A **subspace** $(Y, \tilde{d})$ of (X, d) is obtained if we take a subset $Y \subseteq X$ and restrict d to $Y \times Y$, thus the metric on Y is the restriction

$$\tilde{d} = d_{Y \times Y}$$

$\tilde{d}$ is called the metric **induced** on Y by d.

(2) Let $x, y \in \mathbb{R}^n$ and $p \geq 1$ be a real number. Then the Minkowski inequality:

$$\left(\sum_{i=1}^{n} |x_i + y_i|^p \right)^{1/p} \leq \left(\sum_{k=1}^{n} |x_k|^p \right)^{1/p} + \left(\sum_{m=1}^{n} |y_m|^p \right)^{1/p} \tag{1.1}$$

(3) Let $x, y \in \mathbb{R}^n$. Then the Cauchy-Schwarz inequality:

$$\sum_{i=1}^{n} |x_i y_i| \leq \sqrt{\sum_{k=1}^{n} |x_k|^2} \cdot \sqrt{\sum_{m=1}^{n} |y_m|^2} \tag{1.2}$$

(4) Let (X, d) be a metric space. Then for $x_1, x_2, \cdots, x_n \in X$ we have

$$d(x_1, x_n) \leq d(x_1, x_2) + d(x_2, x_3) + \cdots + d(x_{n-1}, x_n)$$

1.2 Basic Principles of Topology

There is a considerable number of auxiliary concepts which play a role in connection with metric spaces. Those which we shall need are included in this section. We first consider important types of subsets of a given metric space (X, d).

Definition 1.2.1. Let $x_0 \in X$ and r be a real number. Then we define the following sets:

i. $B(x_0, r) = \{x \in X : d(x, x_0) < r\}$ **(Ball with center x_0 and radius r).**

ii. $D(x_0, r) = \{x \in X : d(x, x_0) \leq r\}$ **(Disk with center x_0 and radius r).**

iii. $S(x_0, r) = \{x \in X : d(x, x_0) = r\}$ **(Sphere with center x_0 and radius r).**

The definition immediately implies that

$$S(x_0, r) = D(x_0, r) - B(x_0, r)$$

$B(x_0, r)$ is often called a **Neighborhood** of x_0.

Example 1.2.2.

Describe the ball, disk and sphere in each of the following metric spaces

(1) Real line $\mathbb{R}$.

$B(x_0, r) = \{x \in X : |x - x_0| < r\} = (x_0 - r, x_0 + r)$ (open interval).

$D(x_0, r) = \{x \in X : |x - x_0| \leq r\} = [x_0 - r, x_0 + r]$ (closed interval).

$S(x_0, r) = \{x \in X : |x - x_0| = r\} = \{x_0 + r, x_0 - r\}$.

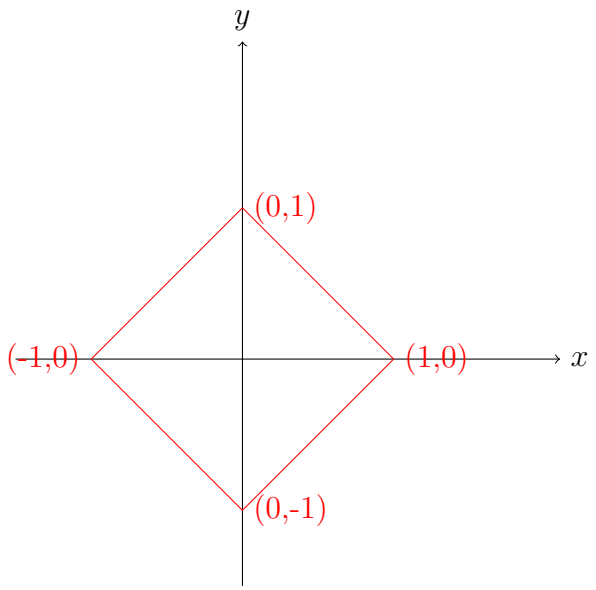
(2) Euclidean plane $\mathbb{R}^2$ with $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$.

$B(y, r) = \left\{x \in \mathbb{R}^2 : \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} < r\right\} = \{x \in \mathbb{R}^2 : (x_1 - y_1)^2 + (x_2 - y_2)^2 < r^2\}$ (inside of the circle with centre y and radius r).

$D(y, r) = \left\{x \in \mathbb{R}^2 : \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \leq r\right\} = \{x \in \mathbb{R}^2 : (x_1 - y_1)^2 + (x_2 - y_2)^2 \leq r^2\}$, (inside of the circle with centre y and radius r as well as its circumference).

$S(y, r) = \left\{x \in \mathbb{R}^2 : \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} = r\right\} = \{x \in \mathbb{R}^2 : (x_1 - y_1)^2 + (x_2 - y_2)^2 = r^2\}$, (circle with centre y and radius r).

(3) Euclidean plane $\mathbb{R}^2$ with $d_1(x, y) = |x_1 - y_1| + |x_2 - y_2|$. Take $x_0 = 0$ and $r = 1$.



$B(0,1) = \{y \in \mathbb{R}^2 : |0 - y_1| + |0 - y_2| < 1\} = \{y \in \mathbb{R}^2 : |y_1| + |y_2| < 1\}$ (The ball will be all the points inside this rhombus (this shape).

$D(0,1) = \{y \in \mathbb{R}^2 : |y_1| + |y_2| \leq 1\}$. The disk will be all the points inside this rhombus (this shape) as well as its perimeter.

$S(0,1) = \{y \in \mathbb{R}^2 : |y_1| + |y_2| = 1\}$. The sphere will be all the points on the perimeter of this rhombus (shape).

Definition 1.2.3 (Open set, closed set)**.**

Let (X, d) be a metric space. A subset $S \subseteq X$ is said to be **open** if for every $x_0 \in S$ there exists $r > 0$ such that the ball $B(x_0, r) \subseteq S$. A subset $M \subseteq X$ is said to be **closed** if its complement (in X) is open, that is, $M^c = X - M$ is open.

Example 1.2.4.

(1) Let (X, d) be a metric space. If $x_0 \in X$ and $r > 0$ then $B(x_0, r)$ is an open set.

(2) Let $X = \mathbb{R}$. For $a \in \mathbb{R}$, show that

$$(a, \infty) = \{x \in \mathbb{R} : x > a\} \quad \text{and} \quad (-\infty, a) = \{x \in \mathbb{R} : x < a\}$$

are open sets.

(3) Show that every disk in $\mathbb{R}^2$ (or in $\mathbb{R}^n$) is not open.

(4) Show that the set of rational numbers is not open in $\mathbb{R}$.

Definition 1.2.5 (Limit point)**.**
Let (X, d) be a metric space. Suppose that $S \subseteq X$, then $x_0 \in X$ (x_0 may or may not be a point of S) is said to be a **limit point** (or an **accumulation point**) for S if every ball with centre x_0 contains at least one point $p \in S$ distinct from x_0. That is for every $B(x_0, r)$

$$(B(x_0, r) - \{x_0\}) \bigcap S \neq \emptyset.$$

We will use the symbol $S^{'}$ to refer to the set of all limit points of S. The **closure** of S consisting of all points of S and the limit points of S, that is

$$\bar{S} = S \cup S^{'}$$

The closure of S is the smallest closed set containing S.

Example 1.2.6. Let $S = (a, b) \subseteq \mathbb{R}$. Show that $S^{'} = [a, b]$.

Let $x \in (a, b)$. Then x is a limit point of $S = (a, b)$. Let $r > 0$. Then the interval $(x - r, x + r)$ will contain points of (a, b) other than x. In fact $(x - r, x + r)$ contains infinitely many points of (a, b) distinct from x. Moreover, a and b are limit points of (a, b). Let $\delta > 0$. Then the interval $(a - \delta, a + \delta)$ contains infinitely many points of (a, b) distinct from a. That is

$$(a - \delta, a + \delta) \setminus \{a\} \bigcap (a, b) \neq \emptyset,$$

thus a is a limit point for (a, b). The same is true for b. Hence $S^{'} = [a, b]$ and $\bar{S} = S \cup S^{'} = (a, b) \cup [a, b] = [a, b]$

Example 1.2.7. Let $S = \left\{1, \frac{1}{2}, \frac{1}{3}, \cdots\right\} \subseteq \mathbb{R}$. Show that $S^{'} = \{0\}$.

For every $\varepsilon > 0$, there exists $n \in \mathbb{N}$ (by Archimedean property) such that $\frac{1}{n} < \varepsilon$. Thus the interval $(-\varepsilon, \varepsilon)$ contains $\frac{1}{n} \in S$ and $\frac{1}{n} \neq 0$. In other words

$$((-\varepsilon, \varepsilon) \setminus \{0\}) \bigcap S \neq \emptyset.$$

This shows that $0 \in S^{'}$.
Now, we have to show that if $x \neq 0$ then $x \notin S^{'}$. Note that if $x < 0$ or $x \geq 1$ then there exists $\varepsilon > 0$ such that

$$((x - \varepsilon, x + \varepsilon) - \{x\}) \cap S = \emptyset.$$

Hence, $x \notin S'$. If $0 < x < 1$ we have two cases, **either** $x \in S$, that is $x = \frac{1}{n}$ for some $n \in \mathbb{N}$. Hence

$$\left((\frac{1}{n+1}, \frac{1}{n-1}) - \{x\}\right) \cap S = \emptyset$$

Thus $x \notin S'$. **Or** $x \notin S$, this implies that $\frac{1}{x} \notin \mathbb{N}$. Hence there exists $n \in \mathbb{N}$ such that $n < \frac{1}{x} < n+1$, that is, $\frac{1}{n+1} < x < \frac{1}{n}$. Hence

$$\left((\frac{1}{n+1}, \frac{1}{n}) - \{x\}\right) \cap S = \emptyset.$$

Therefore $x \notin S'$ and 0 is only the limit point for S.

Example 1.2.8. Let $\mathbb{Q}$ be the set of rational numbers in $\mathbb{R}$. Then $\mathbb{Q}' = \mathbb{R}$ and $\overline{\mathbb{Q}} = \mathbb{Q} \cup \mathbb{Q}' = \mathbb{R}$.

Suppose that $x_0 \in \mathbb{R}$ then for each open interval around x_0 contains many rational numbers (by the density of the rational numbers). Thus $\mathbb{Q}' = \mathbb{R}$. Hence $\overline{\mathbb{Q}} = \mathbb{Q} \cup \mathbb{Q}' = \mathbb{R}$.

Example 1.2.9. Let S be a finite set in $\mathbb{R}^n$. Then $S' = \emptyset$.

Solution: Homework

Example 1.2.10. Let $\mathbb{Z}$ be the set of all integer numbers in $\mathbb{R}$. Then $\mathbb{Z}' = \emptyset$.

Solution:

The same as the previous example.

Theorem 1.2.11. *Let X be a metric space (or a Topological space). Then $S \subseteq X$ is closed if and only $S' \subseteq S$. That is $S \subseteq X$ is closed if and only if it contains all its limit points. (or S is closed if and only if $S = \bar{S}$).*

Proof. ($\Rightarrow$) Suppose that S is closed, then by the definition its complement $X \backslash S$ is open. Take $x \notin S$, that is $x \in X \backslash S$. Since is open then there is $r > 0$ such that the ball $B(x, r) \subseteq X \backslash S$. Hence $B(x, r) \cap S = \emptyset$, this implies that x is not a limit point for S. Thus $S' \subseteq S$.

($\Leftarrow$) For the converse, suppose that $S' \subseteq S$. To prove S is closed we will show that $X \backslash S$ is open. Let $x \in X \backslash S$. Since x is not a limit point for S then there exists $r > 0$ such that $B(x, r) \cap S = \emptyset$. That is $B(x, r) \subseteq X \backslash S$. Hence $X \backslash S$ is open and S is closed. □

1.3 Convergence in Metric Spaces

Definition 1.3.1. Let (X, d) be a metric space and $\{a_n\}_{n=1}^{\infty}$ be a sequence in X. We say that $\{a_n\}_{n=1}^{\infty}$ **converges** to $a_0 \in X$ as $n \to \infty$, if for any $\varepsilon > 0$ there exists an integer $k \in \mathbb{N}$ such that

$$d(a_n, a_0) < \varepsilon \qquad \text{for all} \quad n \geq k.$$

a_0 is called the convergence point for the sequence $\{a_n\}_{n=1}^{\infty}$ and we write

$$a_n \to a_0 \qquad \text{or} \qquad \lim_{n\to\infty} a_n = a_0.$$

In other words, $\{a_n\}_{n=1}^{\infty}$ converges to a_0 if for every ball with centre a_0 contains most of the terms of $\{a_n\}_{n=1}^{\infty}$ with exclusion of finite number of terms.

Lemma 1.3.2. *Let (X, d) be a metric space. Then:*

1. *The limit of a convergent sequence in X is unique.*

2. *A convergent sequence in X is bounded.*

3. *If $a_n \to a_0$ and $b_n \to b_0$ in X, then $d(a_n, b_n) \to d(a_0, b_0)$.*

Theorem 1.3.3. *Let S be a non-empty subset of a metric space (X, d) and $\bar{S}$ be the closure of S. Then:*

1. *$x_0 \in \bar{S}$ if and only if there is a sequence $\{x_n\}_{n=1}^{\infty}$ in S such that $x_n \to x_0$.*

This means that if $x_0 \in S$ or x_0 is a limit point for S ($x_0 \in S'$), then there exists a sequence $x_n \in S$ such that $x_n \to x_0$ and conversely, if x_n is a sequence in S with $x_n \to x_0$ then $x_0 \in S$ or $x_0 \in S'$.

2. *S is closed if and only the case $x_n \in S$ with $x_n \to x_0$ implies that $x_0 \in S$.*

Cauchy Sequences (Fundamental Sequences)

Definition 1.3.4. Let (X, d) be a metric space. Then the sequence $\{a_n\}_{n=1}^{\infty}$ is called a **Cauchy sequence** if for each $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that

$$d(a_n, a_m) < \varepsilon \qquad \text{for all} \quad n, m > k.$$

Theorem 1.3.5. *Let (X, d) be a metric space. Then every convergent sequence in X is a Cauchy sequence.*

Proof. Let $\{a_n\}_{n=1}^{\infty}$ be a convergent sequence in X such that $a_n \to a_0 \in X$. That is for each $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that

$$d(a_n, a_0) < \frac{\varepsilon}{2} \quad \text{for all} \quad n \geq k.$$

Thus for all $n, m \geq k$

$$d(a_n, a_m) \leq d(a_n, a_0) + d(a_m, a_0) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequences. □

The converse of this theorem is not true. That is if $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence in a metric space (X, d) then $\{a_n\}_{n=1}^{\infty}$ may not converge in X. Consider the following examples:

Example 1.3.6.

(1) Let $X = \mathbb{R}\setminus\{0\}$ and $d(x, y) = |x - y|$ for all $x, y \in X$. It is clear that (X, d) is a metric space. Suppose that $a_n := \frac{1}{n}$ for all $n \in \mathbb{N}$, that is $a_n = \left\{1, \frac{1}{2}, \frac{1}{3}, \cdots\right\}$. Then a_n is a Cauchy sequence (Why? show that) and $a_n \to 0$ but $0 \notin X$. Hence a_n does not converge in X.

(2) Let $X = \mathbb{Q}$ and $d(x, y) = |x - y|$ for all $x, y \in X$. It is clear that (X, d) is a metric space. We have noted that for every real number r there exists a Cauchy sequence of rational numbers r_n such that $r_n \to r$. Suppose that $r = \sqrt{2}$, thus $r_n \to \sqrt{2}$. But $\sqrt{2} \notin X$, thus r_n does not converge in X.

Complete Metric Spaces

Definition 1.3.7. Let (X, d) be a metric space. Then (X, d) is called a **complete metric space** if every Cauchy sequence in X is convergent in X.

From that previous example we note that $(\mathbb{R}\setminus\{0\}, d)$ and $(\mathbb{Q}, d)$ are not complete metric spaces.

Example 1.3.8. Prove that the Euclidean space $(\mathbb{R}^2, d)$ is complete metric space.

Proof. Let $\{a_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $\mathbb{R}^2$ and let $\varepsilon > 0$. Suppose that $a_n = (x_n, y_n)$ for all $n \in \mathbb{N}$, where $x_n, y_n \in \mathbb{R}$. Since $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence then there exists $k \in \mathbb{N}$ such that

$$d(a_n, a_m) < \varepsilon \quad \text{for all } n, m \geq k$$

that is

$$\sqrt{(x_n - x_m)^2 + (y_n - y_m)^2} < \varepsilon$$

$$(x_n - x_m)^2 + (y_n - y_m)^2 < \varepsilon^2$$

This implies that

$$|x_n - x_m| < \frac{\varepsilon}{\sqrt{2}} \quad \text{and} \quad |y_n - y_m| < \frac{\varepsilon}{\sqrt{2}} \quad \text{for all } n, m \geq k.$$

That is $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are Cauchy sequences in $\mathbb{R}$. Since $\mathbb{R}$ is a complete metric space then there exists $x_0, y_0 \in \mathbb{R}$ such that

$$x_n \to x_0 \quad \text{and} \quad y_n \to y_0,$$

That is there exists $k \in \mathbb{N}$ such that

$$|x_n - x_0| < \frac{\varepsilon}{2} \quad \text{and} \quad |y_n - y_0| < \frac{\varepsilon}{2} \quad \text{for all } \ n \geq k.$$

Set $a_0 = (x_0, y_0)$, then

$$\begin{aligned}[d(a_n, a_0)]^2 &= (x_n - x_0)^2 + (y_n - y_0)^2 \\ &< \frac{\varepsilon^2}{4} + \frac{\varepsilon^2}{4} = \frac{\varepsilon^2}{2} \quad \text{for all } \ n \geq k,\end{aligned}$$

hence $d(a_n, a_0) < \varepsilon$ for all $n > k$, thus $a_n \to a_0$ and $\mathbb{R}^2$ is a complete metric space. □

Lemma 1.3.9. *Let (X, d) be a complete metric space. Suppose that S is a non-empty closed subset of X, then (S, d) is complete metric space.*

1.4 Vector Spaces

In this section, the definition of a vector space is given and some examples.

Recall that a field $\mathbb{F}$ is a set of numbers with the property that if $\alpha, \beta \in \mathbb{F}$, then $\alpha + \beta, \alpha - \beta, \alpha\beta$ and $\frac{\alpha}{\beta}$ are also in $\mathbb{F}$ with $\beta \neq 0$ in the expression $\frac{\alpha}{\beta}$).

Definition 1.4.1. A vector space consists of a non-empty set V (elements of V are called vectors), a field $\mathbb{F}$ (elements of $\mathbb{F}$ are called scalars), and two operations. An operation called vector addition that takes two vectors $v, w \in V$, and produces a third vector, written $v + w \in V$. Another operation called

scalar multiplication that takes a scalar $c \in \mathbb{F}$ and a vector $v \in V$, and produces a new vector, written $cv \in V$.

These two operations satisfy the following conditions (called axioms):

(1) Associativity of vector addition: $(u + v) + w = u + (v + w)$ for all $u, v, w \in V$.

(2) Existence of a zero vector: There is a vector in V, written 0 and called the zero vector, which has the property that $u + 0 = u$ for all $u \in V$.

(3) Existence of negatives: For every $u \in V$, there is a vector in V , written $-u$ and called the negative of u, which has the property that $u + (-u) = 0$.

(4) Associativity of multiplication: $(ab)u = a(bu)$ for any $a, b \in \mathbb{F}$ and $u \in V$.

(5) Distributivity: $(a + b)u = au + bu$ and $a(u + v) = au + av$ for all $a, b \in \mathbb{F}$ and $u, v \in V$.

(6) Unitarity: $1u = u$ for all $u \in V$.

Consider the following examples:

Example 1.4.2. The set $\mathbb{R}^n$ is a real vector space with the two algebraic operations defined as follows:

$$x + y = (x_1 + y_1, x_2 + y_2, ..., x_n + y_n),$$

$$\alpha x = (\alpha x_1, \alpha x_2, ..., \alpha x_n), (\alpha \in \mathbb{R}).$$

Example 1.4.3. The set $\mathbb{C}^n$ is a complex vector space with the two algebraic operations defined as follows:

$$z + w = (z_1 + w_1, z_2 + w_2, ..., z_n + w_n),$$

$$\alpha z = (\alpha z_1, \alpha z_2, ..., \alpha z_n), (\alpha \in \mathbb{C}).$$

Example 1.4.4. The set $C[a, b]$ of all real valued functions which are defined and continuous on a given closed interval $J = [a, b]$ is a real vector space with the algebraic operations defined for each $f, g \in C[a, b]$, as follows:

$$(f + g)(t) = f(t) + f(t), \quad \text{for all } t \in [a, b]$$

$$(\alpha f)(t) = \alpha f(t), (\alpha \in \mathbb{R}).$$

Example 1.4.5. The set $\ell^2(\mathbb{C})$, $1 \leq p < \infty$ of all complex sequences $a_n = (a_1, a_2, a_3, \cdots)$ for which

$$\sum_{n=1}^{\infty} |a_n|^p < \infty$$

is a vector space with the algebraic operations defined as usual in connection with sequences, that is,

$$a_n + b_n = (a_1 + b_1, a_2 + b_2, a_3 + b_3, \ldots)$$

$$\alpha a_n = (\alpha a_1, \alpha a_2, \alpha a_3, \ldots)$$

Chapter 2

Hilbert Spaces

2.1 Normed Spaces

In this section we recall the definition of normed spaces with some examples.

Definition 2.1.1. A normed space is a vector space N such that for each vector $x \in N$, there is defined a non-negative real number called the norm of x, denoted $\|x\|$, satisfying the following conditions:

(1) $\|x\| \geq 0$ for all $x \in N$.

(2) $\|x\| = 0$ if and only if $x = \theta$.

(3) $\|x + y\| \leq \|x\| + \|y\|$, for all $x, y \in N$.

(4) $\|\lambda x\| = |\lambda| \cdot \|x\|$, for all $x \in N$ and any scalar λ.

It is clear from (4) that $\|-x\| = \|ix\| = \|x\|$.

Remark 2.1.2. Every subspace of a normed space is itself a normed space with respect to the same norm.

Consider the following examples:

Example 2.1.3. The vector space $\mathbb{R}$ of real numbers is a normed space with respect to the norm $\|x\| = |x|$ for all $x \in \mathbb{R}$.

Let $x, y, z \in \mathbb{R}$ and α be a scalar. Then

(1) It is clear that $\|x\| \geq 0$, for all $x \in \mathbb{R}$.

(2)

$$\|x\| = 0 \Longleftrightarrow |x| = 0 \Longleftrightarrow x = \theta.$$

(3)

$$\|\alpha x\| = |\alpha x| = |\alpha|\,|x| = |\alpha|\,\|x\|$$

(4)

$$\|x + y\| = |x + y| \leq |x| + |y| = \|x\| + \|y\|$$

Example 2.1.4. The vector space $\mathbb{C}$ of complex numbers is a normed space with respect to the norm $\|z\| = |z| = \sqrt{z_1^2 + z_2^2}$ for all $z \in \mathbb{C}$.

Solution:Homework

Example 2.1.5. The vector space $\mathbb{R}^n$; of all n-tuples $x = (x_1, x_2, ..., x_n)$ of real numbers is a normed space with respect to the norm

$$\|x\| = \left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p},$$

where $1 \leq p < \infty$.

Let $x, y \in \mathbb{R}^n$ and α be a scalar. Then

(1) It is clear that $\|x\| \geq 0$, for all $x \in \mathbb{R}^n$.

(2)

$$\|x\| = 0 \Longleftrightarrow \left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p} = 0 \Longleftrightarrow \sum_{k=1}^{n} |x_k|^p = 0 \Longleftrightarrow x_k = 0, \text{ for all } k = 1, 2, ..., n.$$

Thus $\|x\| = 0 \Longleftrightarrow x = 0$.

(3)

$$\|x + y\| = \left(\sum_{k=1}^{n} |x_k + y_k|^p\right)^{1/p}$$

$$\leq \left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p} + \left(\sum_{k=1}^{n} |y_k|^p\right)^{1/p},$$

by Minkowski inequality for sums. Hence $\|x + y\| \leq \|x| + \|y\|$

(4)

$$\|\alpha x\| = \left(\sum_{k=1}^{n} |\alpha x_k|^p\right)^{1/p} = \left(|\alpha|^p \sum_{k=1}^{n} |x_k|^p\right)^{1/p} = |\alpha|\,\|x\|\,.$$

Example 2.1.6. The vector space $\ell_p(\mathbb{C})$ of all sequences of complex numbers for which

$$\sum_{n=1}^{\infty} |x_n|^p < \infty,$$

where, $1 \leq p < \infty$, is a normed space with the following norm:

$$\|x\| = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p},$$

Let $x, y \in \ell_p(\mathbb{C})$ and α be a scalar. Then

(1) It is clear that $\|x\| \geq 0$, for all $x \in \ell_p(\mathbb{C})$.

(2)

$$\|x\| = 0 \Longleftrightarrow \left(\sum_{k=1}^{\infty} |x_k|^p \right)^{1/p} = 0 \Longleftrightarrow \sum_{k=1}^{\infty} |x_k|^p = 0 \Longleftrightarrow x_k = 0, \text{ for all } k = 1, 2, ..., n.$$

Thus $\|x\| = 0 \Longleftrightarrow x = 0$.

(3)

$$\|x + y\| = \left(\sum_{k=1}^{\infty} |x_k + y_k|^p \right)^{1/p}$$

$$\leq \left(\sum_{k=1}^{\infty} |x_k|^p \right)^{1/p} + \left(\sum_{k=1}^{\infty} |y_k|^p \right)^{1/p},$$

by Minkowski inequality for sums. Hence $\|x + y\| \leq \|x| + \|y\|$.

(4)

$$\|\alpha x\| = \left(\sum_{k=1}^{\infty} |\alpha x_k|^p \right)^{1/p} = \left(|\alpha|^p \sum_{k=1}^{\infty} |x_k|^p \right)^{1/p} = |\alpha| \, \|x\| \, .$$

Example 2.1.7. The vector space $C[a, b]$, of all real valued continuous functions defined on the interval $[a, b]$, is a normed space with respect to the following norm:

$$\|f\| = \max_{a \leq x \leq b} |f(x)|, \quad \text{for all } f \in C[a, b]. \tag{2.1}$$

Let $f, g \in C[a, b]$ and α be a scalar. Then

(1) It is clear that $\|f\| \geq 0$, for all $f \in C[a,b]$.

(2)

$$\|f\| = 0 \Longleftrightarrow \max_{a\leq x\leq b} |f(x)| = 0 \Longleftrightarrow |f(x)| = 0, \text{ for all } x \in [a,b].$$

Thus $\|f\| = 0 \Longleftrightarrow f$ is the zero function..

(3)

$$\begin{aligned}\|f+g\| &= \max_{a\leq x\leq b} |f(x)+g(x)| \\ &\leq \max_{a\leq x\leq b} \{|f(x)|+|g(x)|\} = \max_{a\leq x\leq b} |f(x)| + \max_{a\leq x\leq b} |g(x)| \\ &= \|f\| + \|g\|\end{aligned}$$

(4)

$$\|\alpha x\| = \max_{a\leq x\leq b} |(\alpha f)(x)| = \max_{a\leq x\leq b} \{|\alpha||f(x)|\} = |\alpha| \max_{a\leq x\leq b} |f(x)| = |\alpha|\,\|f\|$$

Example 2.1.8. The vector space $C[a,b]$, of all real valued continuous functions defined on the interval $[a,b]$, is a normed space with respect to the following norm

$$\|f\|_1 = \int_a^b |f(x)|\,dx, \qquad \text{for all } f \in C[a,b]. \tag{2.2}$$

Solution: Homework

Theorem 2.1.9. *Let $(N, \|.\|)$ be a normed space, then (N,d) is a metric space, where $d(x,y) = \|x-y\|$ for all $x, y \in N$.*

Proof. Let $x, y, z \in N$, then

(1) It is clear that $d(x,y) = \|x-y\| \geq 0$ and finite.

(2) $d(x,x) = 0 \Longleftrightarrow \|x-x\| = 0 \Longleftrightarrow x - x = \theta \Longleftrightarrow x = x$

(3) $d(x,y) = \|x-y\| = \|(-1)(y-x)\| = |-1|\,\|y-x\| = \|y-x\| = d(y,x)$

(4) $$\begin{aligned} d(x,y) = \|x-y\| &= \|x-z+z-y\| \\ &= \|(x-z)+(z-y)\| \\ &\leq \|x-z\| + \|z-y\| \end{aligned}$$

$$= d(x,z) + d(z,y).$$

Hence, $d(x,y) \leq d(x,z) + d(z,y)$.

Therefore, (N,d) is a metric space. □

We end this section by the following example, which shows that the converse of the previous theorem is not true in general,

Example 2.1.10. Let X_s be the set of all sequences of complex numbers and let d be defined by

$$d(x,y) = \sum_{i=1}^{\infty} \frac{|xi - yi|}{2^i \left(1 + |xi - yi|\right)}$$

where, $x = (x_1, x_2, x_3, \ldots), y = (y_1, y_2, y_3, \ldots) \in X_s$.

Then (X_s, d) is a metric space.

For any $x, y, z \in X_s$ we have:

(1)

$|xi - yi| \geq 0$, $\forall i = 1, 2, 3, \ldots$.

Thus

$$\frac{|xi - yi|}{1 + |xi - yi|} \geq 0 \Longrightarrow \frac{|xi - yi|}{2^i \left(1 + |xi - yi|\right)} \geq 0, \quad \text{for all } i = 1, 2, 3, \ldots$$

Hence,

$$\sum_{i=1}^{\infty} \frac{|xi - yi|}{2^i \left(1 + |xi - yi|\right)} \geq 0 \Longrightarrow d(x,y) \geq 0, \quad \text{for all } x, y \in X_s$$

(2)

$$d(x,y) = 0 \Longleftrightarrow \sum_{i=1}^{\infty} \frac{|xi - yi|}{2^i \left(1 + |xi + yi|\right)} = 0 \Longleftrightarrow \frac{|xi - yi|}{2^i \left(1 + |xi + yi|\right)} = 0, \text{ for all } i$$

$$\Longleftrightarrow |xi - yi| = 0, \quad \text{for all } i$$

$$\Longleftrightarrow xi - yi = 0, \quad \text{for all } i$$

$$\Longleftrightarrow x = y.$$

(3)

$$d(x,y) = \sum_{i=1}^{\infty} \frac{|xi - yi|}{2^i \left(1 + |xi - yi|\right)} = \sum_{i=1}^{\infty} \frac{|yi - xi|}{2^i \left(1 + |yi - xi|\right)} = d(y,x).$$

(4)

$$|x_i - z_i| = |x_i - y_i + y_i - z_i| \Longrightarrow |x_i - z_i| \leq |x_i - y_i| + |y_i - z_i|, \text{ for all } i,$$

thus

$$\frac{1}{|x_i - z_i|} \geq \frac{1}{|x_i - y_i| + |y_i - z_i|} \Longrightarrow \frac{1}{|x_i - z_i|} + 1 \geq \frac{1}{|x_i - y_i| + |y_i - z_i|} + 1, \text{ for all } i$$

so,

$$\frac{1 + |x_i - z_i|}{|x_i - z_i|} \geq \frac{1 + |x_i - y_i| + |y_i - z_i|}{|x_i - y_i| + |y_i - z_i|} \Longrightarrow \frac{|x_i - z_i|}{1 + |x_i - z_i|} \leq \frac{|x_i - y_i| + |y_i - z_i|}{1 + |x_i - y_i| + |y_i - z_i|}, \text{ for all } i$$

hence,

$$\frac{|x_i - z_i|}{1 + |x_i - z_i|} \leq \frac{|x_i - y_i|}{1 + |x_i - y_i| + |y_i - z_i|} + \frac{|y_i - z_i|}{1 + |x_i - y_i| + |y_i - z_i|} \leq \frac{|x_i - y_i|}{1 + |x_i - y_i|} + \frac{|y_i - z_i|}{1 + |y_i - z_i|}, \text{ for all } i$$

therefore,

$$\frac{|x_i - z_i|}{2^i(1 + |x_i - z_i|)} \leq \frac{|x_i - y_i|}{2^i(1 + |x_i - y_i|)} + \frac{|y_i - z_i|}{2^i(1 + |y_i - z_i|)}, \text{ for all } i$$

thus

$$\sum_{i=1}^{\infty} \frac{|x_i - z_i|}{2^i\,(1 + |x_i - z_i|)} \leq \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{2^i\,(1 + |x_i - y_i|)} + \sum_{i=1}^{\infty} \frac{|y_i - z_i|}{2^i\,(1 + |y_i - z_i|)}$$

hence,

$$d(x, z) \leq d(x, y) + d(y, z).$$

Therefore, d is a matric on X_s and (X_s, d) is a metric space.

This metric space can not be normed space, because if there is norm such that $d(x, y) = \|x - y\|$, then

$$d(\alpha x, \alpha y) = \|\alpha x - \alpha y\| = |\alpha|\,\|x - y\| = |\alpha|\,d(x, y),$$

must satisfied. However, for the metric under consideration, this relation is not valid as:

$$d(\alpha x, \alpha y) = \sum_{i=1}^{\infty} \frac{|\alpha|\,|xi - yi|}{2^i\,(1 + |\alpha|\,|xi - yi|)}$$

and

$$|\alpha|\,d(x, y) = |\alpha| \sum_{i=1}^{\infty} \frac{|xi - yi|}{2^i\,(1 + |xi - yi|)}$$

2.2 Convergence in Normed Spaces

This section is concerned with sequences in normed spaces.

Definition 2.2.1. Let $(N, \|.\|)$ be a normed space and $\{x_n\}_{n=1}^{\infty}$ be a sequence in N. Then $\{x_n\}_{n=1}^{\infty}$ is called convergent to $x_0 \in N$ if for all $\epsilon > 0$ there exists $k \in \mathbb{N}$ such that

$$\|x_n - x_0\| < \epsilon, \quad \text{for all } n \geq k.$$

Definition 2.2.2. Let $(N, \|.\|)$ be a normed space and $\{x_n\}_{n=1}^{\infty}$ be a sequence in N. Then $\{x_n\}_{n=1}^{\infty}$ is called Cauchy sequence if for all $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that

$$\|x_n - x_m\| < \epsilon, \quad \text{for all } n, m \geq k.$$

Recall that a subset S in a normed space N is bounded if and only if there is a positive number M such that $\|x\| \leq M$ for all $x \in S$.

As in metric spaces the sequences in normed spaces have the following properties,

Theorem 2.2.3.

Let $(N, \|.\|)$ be a normed space $\{x_n\}_{n=1}^{\infty}$ be a sequence in N. Then

(1) *The limit of every convergent sequence in N is unique.*

(2) *Every convergent sequence is bounded.*

(3) *Every Cauchy sequence is bounded.*

(4) *Every convergent sequence is Cauchy sequence.*

Remark 2.2.4. *Let $(N, \|.\|)$ be a normed space, not every Cauchy sequence in N is convergent in N.*

Definition 2.2.5. Let $(N, \|.\|)$ be a normed space. If N is complete metric space with the metric

$$d(x, y) = \|x - y\|, \quad \text{for all } x, y \in N,$$

then $(N, (\|.\|))$ is called a Banach space.

Remark 2.2.6. *Let $(N, \|.\|)$ be a Banach space (complete normed space). If S is a subspace of N, the S may not complete.*

Theorem 2.2.7. *A subspace S of a Banach space $(N, \|.\|)$ is complete if and only if S is closed.*

Proof. Homework. □

Example 2.2.8. The normed space $(\mathbb{R}, \|x\| = |x|)$ is a Banach space.

Example 2.2.9. The normed space $(\mathbb{C}, \|z\| = \sqrt{z_1^2 + z_2^2})$ is a Banach space.

Example 2.2.10. The normed space $(\mathbb{R}^n, \|x\|)$, where

$$\|x\| = \left(\sum_{k=1}^{n} |x_k|^p \right)^{1/p},$$

is a Banach space.

Example 2.2.11. The normed space $(\ell_p(\mathbb{C}), \|x\|)$ of all sequences of complex numbers (as in Example 2.1.6) where,

$$\|x\| = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p},$$

is a Banach space.

Example 2.2.12. The normed space $C[a, b]$, of all real valued continuous functions is defined on the interval $[a, b]$, with the following two norms:

$$\|f\| = \max_{a \le x \le b} |f(x)|, \quad \text{for all } f \in C[a, b]. \tag{2.3}$$

and

$$\|f\|_1 = \int_a^b |f(x)|\, dx, \quad \text{for all } f \in C[a, b]. \tag{2.4}$$

is a Banach space with respect to the norm in Equation 2.3, however it is not a Banach space with respect to the norm in Equation 2.4.

Remark 2.2.13. Theorem 2.1.10, shows that every normed space is a metric space, thus we can conclude that every Banach space is a complete metric space.

2.3 Inner Product Spaces or Pre-Hilbert Spaces

Definition 2.3.1. Let X be a vector space over the real or complex numbers. The inner product for any two vectors $x, y \in X$ is a mapping denoted by $\langle x, y \rangle$, defined on $X \times X$ into $\mathbb{R}$ or $\mathbb{C}$, if the following conditions are satisfied for all $x, y, z \in X$ and for any scalar α.

.

(1) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$.

(2) $\langle \alpha x, y\rangle = \alpha \langle x, y\rangle$.

(3) $\langle x, y\rangle = \overline{\langle y, x\rangle}$.

(4) $\langle x, x\rangle \geq 0, \forall x \in X$ and $\langle x, x\rangle = 0$ if and only if $x = \theta$

The vector space X with the inner product $\langle x, y\rangle$ is denoted by $(X, \langle ., .\rangle)$ and called product space of pre-Hilbert space.

Remark 2.3.2. *Let $(X, \langle ., .\rangle)$ be an inner product space. Then for each $x, y, z \in X$ and for any scalar β, we have*

(1) *$\langle x, y\rangle$ denotes the conjugate of the number $\langle y, x\rangle$. Therefore, if $\langle x, y\rangle$ is real, then $\langle x, y\rangle = \langle y, x\rangle$.*

(2) *$\langle x, x\rangle$ must be real.*

In (3) $\langle x, y\rangle = \overline{\langle y, x\rangle}$, thus $\langle x, x\rangle = \overline{\langle x, x\rangle}$. This implies that $\langle x, x\rangle$ is real number.

(3) *The function $\langle ., .\rangle : X \times X \to \mathbb{R}(or\mathbb{C})$ is linear in the first variable (condition (1) and (2)).*

It is also linear in the second variable.

a) $\langle x, y + z\rangle = \langle x, y\rangle + \langle x, z\rangle$.

b) $\langle x, \beta y\rangle = \overline{\beta} \langle x, y\rangle$

Proof. **(a)**

$$\langle x, y + z\rangle = \overline{\langle y + z, x\rangle} \quad \text{using (3) from definition 2.3.1,}$$

we also have

$$\langle y + z, x\rangle = \langle y, x\rangle + \langle z, x\rangle \quad \text{using (1) from definition 2.3.1,}$$

thus

$$\langle x, y + z\rangle = \overline{\langle y, x\rangle + \langle z, x\rangle} = \overline{\langle y, x\rangle} + \overline{\langle z, x\rangle},$$

hence

$$\langle x, y + z\rangle = \langle x, y\rangle + \langle x, z\rangle \quad \text{using (3) from definition 2.3.1.}$$

□

Proof. **(b)**

$$\langle x, \beta y\rangle = \overline{\langle \beta y, x\rangle} \quad \text{using (3) from definition 2.3.1,}$$

since,

$$\langle \beta y, x\rangle = \beta \langle y, x\rangle \quad \text{using (2) from definition 2.3.1,}$$

therefore

$$\langle x, \beta y\rangle = \overline{\langle \beta y, x\rangle} = \overline{[\beta \langle y, x\rangle]} = \overline{\beta}\ \overline{\langle y, x\rangle},$$

hence,

$$\langle x, \beta y\rangle = \overline{\beta}\langle x, y\rangle, \quad \text{using (3) from definition 2.3.1.}$$

□

(4) *For any $x, y \in X$, $\langle x, \theta\rangle = \langle \theta, y\rangle = 0$*

Proof. **(4)** (Homework)

□

(5)

$$\left\langle \sum_{k=1}^{n} \alpha_k x_k, \sum_{m=1}^{n} \beta_m, y_m \right\rangle = \sum_{k=1}^{n}\sum_{m=1}^{n} \alpha_k \overline{\beta}_m \langle x_k, y_m\rangle .$$

Proof. (Homework)

□

(6)

$$\langle x - y, z\rangle = \langle x, z\rangle - \langle y, z\rangle .$$

Proof. (Homework)

□

(7)

$$\langle x, y - z\rangle = \langle x, y\rangle - \langle x, z\rangle .$$

Proof. (Homework)

□

(8) *If $\langle x, z\rangle = \langle y, z\rangle$ for all $z \in X$, the $x = y$.*

Proof. (Homework)

□

Now, we give some examples of inner product spaces (pre-Hilbert spaces)

Example 2.3.3. The Euclidean plane $\mathbb{R}^n$ of all n-tuples $x = (x_1, x_2, x_3, \cdots, x_n)$ of real numbers is an inner product space (pre-Hilbert space) with the following inner product:

$$\langle x, y\rangle = \sum_{k=1}^{n} x_k y_k$$

where $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n, y = (y_1, y_2, ..., y_n) \in \mathbb{R}^n$.

Solution: Let $x, y, z \in \mathbb{R}^n$ and α be a scalar. Then

(1)

$$\begin{aligned}\langle x + y, z\rangle &= \sum_{k=1}^{n}(x_k + y_k)z_k,\\ &= \sum_{k=1}^{n} x_k z_k + \sum_{k=1}^{n} y_k z_k,\\ &= \langle x, z\rangle + \langle y, z\rangle .\end{aligned}$$

(2)

$$\langle \alpha x, y\rangle = \sum_{k=1}^{n} \alpha x_k y_k = \alpha \sum_{k=1}^{n} x_k y_k = \alpha \langle x, y\rangle .$$

(3)

$$\langle x, y\rangle = \sum_{k=1}^{n} x_k y_k = \sum_{k=1}^{n} y_k x_k = \overline{\sum_{k=1}^{n} y_k x_k} \quad \text{(since } x_k y_k \text{ is real number for each k)}$$

thus $\langle x, y\rangle = \overline{\langle y, x\rangle}$

(4)

$$\langle x, x\rangle = \sum_{k=1}^{n} x_k x_k = \sum_{k=1}^{n} x_k^2.$$

It clear that $\langle x, x\rangle \geq 0$.

and

$$\langle x, x\rangle = 0 \Longleftrightarrow \sum_{k=1}^{n} x_k^2 = 0 \Longleftrightarrow x_1 = 0, x_2 = 0, \cdots x_n = 0.$$

Thus $\langle x, x\rangle = 0 \Longleftrightarrow x = \theta$ is the zero vector.

Example 2.3.4. The vector space $\mathbb{C}^n$ of all n-tuples $z = (z_1, z_2, \cdots, z_n)$ of complex numbers is an inner product space with the inner product

$$\langle z, w\rangle = \sum_{k=1}^{n} z_k \overline{w}_k, \text{ for all } z, w \in \mathbb{C}^n,$$

Solution: (Homework)

Example 2.3.5. For $p = 2$, the space $\ell_p(\mathbb{C})$ was defined in Example (1.4.5) is an inner product space (Pre-Hilbert space) with inner product given by:

$$< x, y >= \sum_{n=1}^{\infty} x_n \bar{y}_n$$

Solution: (Homework)

Example 2.3.6. *Let $C[a, b]$ be the vector space of all real valued continuous functions defined on $[a, b]$. Then $C[a, b]$ is pre-Hilbert space with the following two inner product:*

(1) $\langle f, g\rangle = \int_a^b f(x)g(x)dt$ for all $f, g \in C[a, b]$.

(2) $\langle f, g\rangle = \int_a^b f(t)g(t)w(t)dt$ for all $f, g \in C[a, b]$, where $w(t) \geq 0 \in C[a, b]$.

Solution: *(Homework)*

Example 2.3.7. Let $1 \leq p < \infty$. The vector space $L^p([a, b])$ of all complex integrable functions f on $[a, b]$, is inner product space with the following inner:

$$\langle f, g\rangle = \int_a^b f(z)\overline{g(z)}dz, \quad \text{for all } f, g \in L^p([a, b])$$

Solution:(Homework)

Lemma 2.3.8. *(Cauchy-Schwartz inequality)*

Let $(X, \langle ., .\rangle)$ be an inner product space. Then for all $x, y \in X$, we have

$$|\langle x, y\rangle|^2 \leq \langle x, x\rangle \langle y, y\rangle$$

Proof. Let $x, y \in X$ if $x = \theta$ or $y = \theta$, then the inequality satisfied.

Suppose that $y \neq \theta$. Let $\lambda = \frac{\langle x,y\rangle}{\langle y,y\rangle} \neq 0$. Then $\langle x - \lambda y, x - \lambda y\rangle \geq 0$.

Also,

$$0 \leq \langle x - \lambda y, x - \lambda y\rangle = \langle x, x - \lambda y\rangle - \lambda \langle y, x - \lambda y\rangle = \langle x, x\rangle - \langle x, \lambda y\rangle - \lambda (\langle y, x\rangle - \langle y, \lambda y\rangle)$$

$$= \langle x, x\rangle - \bar{\lambda} \langle x, y\rangle - \lambda \langle y, x\rangle + |\lambda|^2 \langle y, y\rangle$$

as, $\lambda = \frac{\langle x,y\rangle}{\langle y,y\rangle}$, then

$$0 \leq \langle x,x\rangle - \frac{\overline{\langle x,y\rangle}}{\langle y,y\rangle}\langle x,y\rangle - \frac{\langle x,y\rangle}{\langle y,y\rangle}\langle y,x\rangle + \frac{|\langle x,y\rangle|^2}{|\langle y,y\rangle|^2}\langle y,y\rangle$$

$$0 \leq \langle x,x\rangle - \frac{|\langle x,y\rangle|^2}{\langle y,y\rangle} - \frac{|\langle x,y\rangle|^2}{\langle y,y\rangle} + \frac{|\langle x,y\rangle|^2}{\langle y,y\rangle}$$

$$0 \leq \langle x,x\rangle - \frac{|\langle x,y\rangle|^2}{\langle y,y\rangle} \Rightarrow |\langle x,y\rangle|^2 \leq \langle x,x\rangle\langle y,y\rangle.$$

□

Theorem 2.3.9. *Every inner product space N is a normed space with respect to the norm*

$$\|x\| = \sqrt{\langle x,x\rangle}, \text{ for all } x \in N.$$

Proof. Let $x, y \in N$ and λ be a scalar. Then

(1) It is clear that $\|x\| \geq 0$ for all $x \in N$.

(2)

$$\|x\| = 0 \Longleftrightarrow |\langle x,x\rangle|^{1/2} = 0 \Longleftrightarrow \langle x,x\rangle = 0.$$

Hence, $\|x\| = 0$ if and only if x is the zero vector.

(3) $\|x+y\|^2 = \langle x+y, x+y\rangle = \langle x,x\rangle + \langle y,y\rangle + \langle x,y\rangle + \langle y,x\rangle \leq \|x\|^2 + \|y\|^2 + 2|\langle x,y\rangle|$.

By the Schwartz inequality,

$$|\langle x,y\rangle| = |\langle y,x\rangle| \leq (\langle x,x\rangle\langle y,y\rangle)^{1/2} = \|x\|\|y\|$$

hence $\|x+y\|^2 \leq \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| = (\|x\| + \|y\|)^2$.

Thus $\|x+y\| \leq \|x\| + \|y\|$.

(4) $\|\lambda x\| = |\langle \lambda x, \lambda x\rangle|^{1/2} = |\lambda|\,|\langle x,x\rangle|^{1/2} = |\lambda|\,\|x\|$.

From 1,2,3,4 we get that $(N, \|.\|)$ is a normed space. □

Remark 2.3.10. *(Parallelogram Law)*

A norm on an inner product space X satisfies the parallelogram equality:

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

for all $x, y \in X$

Proof. Since X is an inner-product space, we have

$$\|x+y\|^2 = \langle x+y, x+y\rangle = \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle$$

and

$$\|x-y\|^2 = \langle x-y, x-y\rangle = \langle x,x\rangle - \langle x,y\rangle - \langle y,x\rangle + \langle y,y\rangle$$

hence,

$$\|x+y\|^2 + \|x-y\|^2 = 2\langle x,x\rangle + 2\langle y,y\rangle = 2\|x\|^2 + 2\|y\|^2.$$

□

The next example shows that the parallelogram equality is not valid for arbitrary norm in a vector space.

Example 2.3.11. Let $C[0,2\pi]$ be the space of all real valued continuous functions on $[0,2\pi]$. Then $C[0,2\pi]$ is comlete normed space (Banach space) with the following norm:

$$\|f\| = \sup_{o\le t\le 2\pi} |f(t)| \quad \text{for all } f \in C[0,2\pi].$$

This norm does not satisfy the parallelogram law.

To show this, take $f(t) = \max\{\sin t, 0\}$ and $g(t) = \max\{-\sin t, 0\}$.

Then: $\|f\| = 1, \|g\| = 1, \|f+g\| = 1$ and $\|f-g\| = 1$,

so, $\|f+g\|^2 + \|f-g\|^2 = 2$, but $2\|f\|^2 + 2\|g\|^2 = 4$,

hence, $\|f+g\|^2 + \|f-g\|^2 \neq 2\|f\|^2 + 2\|g\|^2$,

Lemma 2.3.12. *Every inner product space X is a metric space with the following metric:*

$$d(x,y) = \sqrt{\langle x-y, x-y\rangle} = \|x-y\|.$$

Definition 2.3.13. An inner product space H is called a **Hilbert space** if the normed space induced by the inner product is a Banach space (complete normed space). That is every Cauchy sequence $a_n \in H$ with respect to the norm induced by the inner product is convergent in H with respect to this norm.

For example: $\mathbb{R}^n$ and $\mathbb{C}^n$ are Hilbert spaces.

Theorem 2.3.14. *A normed space X is an inner product space if and only if its norm satisfies the parallelogram law.*

Proof. Suppose that X be an inner product space. Then X is a normed space, where: $\|x\| = \sqrt{\langle x, x\rangle}$.

It is easy to check that this norm satisfies the parallelogram law (see remark 2.3.10).

Conversely, suppose that X is a norm space and its norm satisfies the parallelogram law.

Define: $Q(x,y) = \frac{1}{4}\left[\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\right]$.

Then $Q(x,y)$ is an inner product on X.

□

Theorem 2.3.15. *A Banach space is Hilbert space if and only of its norm satisfies the parallelogram law.*

Proof. Suppose that H is a Hilbert space, that is H is complete inner product space.

Then H is a normed space where: $\|x\| = \sqrt{\langle x, x\rangle}$.

It is easy to check that this norm satisfies the parallelogram law (see remark 2.3.10).

Thus H is Banach space.

Conversely, suppose that H is Banach space such that its norm satisfies the parallelogram law. Since, H is normed space, then by theorem (2.3.14)

$Q(x,y) = \frac{1}{4}\left[\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\right]$

is an inner product on H.

Let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence in H, then we need to show that this sequence is convergent in H. Thus H is Hilbert space.

Lemma 2.3.16. *Let $(X.\langle .,.\rangle)$ be an inner product space (pre-Hilbert space). The for every $x, y \in X$, we have:*

$$\langle x, y\rangle = \frac{1}{4}\left[\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\right]$$

Proof. (Homework)

□

□

Example 2.3.17. The Euclidean plane $\mathbb{R}^n$ of all n-tuples $x = (x_1, x_2, x_3, \cdots, x_n)$ of real numbers is

Hilbert space with the following inner product:

$$\langle x, y\rangle = \sum_{k=1}^{n} x_k y_k$$

where $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n, y = (y_1, y_2, ..., y_n) \in \mathbb{R}^n$.

Example 2.3.18. The vector space $\mathbb{C}^n$ of all n-tuples $z = (z_1, z_2, \cdots, z_n)$ of complex numbers is an inner product space with the inner product

$$\langle z, w\rangle = \sum_{k=1}^{n} z_k \overline{w}_k, \text{ for all } z, w \in \mathbb{C}^n,$$

Example 2.3.19. The space $\ell_p(\mathbb{C})$ was defined in Example (1.4.5) is not Hilbert space for $p \neq 2$.

Solution:

Let $x = (1, 1, 0, 0, \cdots)$ and $y = (1, -1, 0, 0, \cdots)$, then $x+y = (2, 0, 0, 0, \cdots)$ and $x-y = (0, 2, 0, 0, \cdots)$

$\|x\| = (\sum_{i=1}^{\infty} |x_i|^p)^{1/p} = 2^{1/p}$ and $\|y\| = (\sum_{i=1}^{\infty} |y_i|^p)^{1/p} = 2^{1/p}$

hence $2\|x\|^2 + 2\|y\|^2$

$\|x + y\| = (2^p)^{1/p} = 2$ and $\|x - y\| = (2^p)^{1/p} = 2$

hence $2\|x\|^2 + 2\|y\|^2 = 2\left(2^{2/p}\right) + 2\left(2^{2/p}\right)$ and $\|x + y\|^2 + \|x - y\|^2 = 2^2 + 2^2 = 8$.

Thus if $p \neq 2$, then $\|x + y\|^2 + \|x - y\|^2 \neq 2\|x\|^2 + 2\|y\|^2$.

Therefore, The space $\ell_p(\mathbb{C})$ is not a Hilbert space for $p \neq 2$ and it is a Hilbert space for $p = 2$.

Example 2.3.20. The space $C[a, b]$ is not a Hilbert space with the norm: $\|f\| = \max_{x\in[a,b]} |f(x)|$.

Solution:Homework

Take $f(x) = 1$ for all $x \in [a, b]$ and $g(x) = \frac{x-a}{b-a}$.

Lemma 2.3.21. *Let $(X, \langle ., .\rangle)$ be an inner product space. Then the inner map is continuous. That is for any two convergent sequences in X, $x_n \to x_0$ and $y_n \to y_0$, then $\langle x_n, y_n\rangle \to \langle x_0, y_0\rangle$.*

Proof. Let $\varepsilon > 0$ and $k \in \mathbb{N}$, then

$$|\langle x_n, y_n\rangle - \langle x_0, y_0\rangle| = |\langle x_n, y_n\rangle - \langle x_n, y_0\rangle + \langle x_n, y_0\rangle - \langle x_0, y_0\rangle|$$

$$\leq |\langle x_n, y_n\rangle - \langle x_n, y_0\rangle| + |\langle x_n, y_0\rangle - \langle x_0, y_0\rangle| = |\langle x_n, y_n - y_0\rangle| + |\langle x_n - x_0, y_0\rangle|$$

since $x_n \to x_0$ as $n \to \infty$ and $y_n \to y_0$ as $n \to \infty$,

then $x_n - x_0 \to 0$ and $n \to \infty$ and $y_n - y_0 \to 0$ and $n \to \infty$

thus $|\langle x_n, y_n\rangle - \langle x_0, y_0\rangle| \to 0$ as $n \to \infty$. □

Theorem 2.3.22. *Let H be a Hilbert space and Y be a subspace of H. Then Y is inner product bspace with the inner that defined on H. Moreover,*

(1) Y is Hilbert space if and only if Y is closed.

(2) If Y is finite dimension, the Y is Hilbert space.

Chapter 3

Orthogonal complements and direct sums

3.1 Orthogonal Vectors

Definition 3.1.1. Let X be a pre-Hilbert space. For two vectors $x, y \in X$ we say that x is orthogonal (perpendicular) on y if $\langle x, y\rangle = 0$. In this case we write $x \perp y$.

Remark 3.1.2. Let X be a pre-Hilbert space.

(1) The orthogonal relation is symmetric, that is if $x \perp y$, then $y \perp x$.

(2) $x \perp \theta$, for all $x \in X$. If $x \perp x$, the $x = \theta$.

(3) A vector $x \in X$ is called arthogonal to a non-empte subset A of X, if $\langle x, y\rangle = 0$ for all $y \in X$. In this case we write $x \perp A$.

(4) Two subsets A, B of X are called orthogonal, $\langle x, y\rangle = 0$ for all $x \in A$ and for all $y \in B$. In this case we write $A \perp B$.

(5) A non-empty set A of X is called orthogonal if $\langle x, y\rangle = 0$ for all $x, y \in X$.

(6) Let A be a no-empty subset of X. Then the set of all vectors in X that orthogonal to A is called the orthogonal complement of A and dented by $A^{\perp}$. That is

$$A^{\perp} = \{x \in X : \langle x, y\rangle = 0 \ \ \forall y \in A\}.$$

Remark 3.1.3. Let X be a pre-Hilbert space. Then

(1) $X^{\perp} = \{\theta\}$ and $\{\theta\}^{\perp} = X$.

(2) If $A \perp B$, the $A \cap B = \theta$.

(3) $A^{\perp}$ is closed sebset of X.

(4) If $A \subseteq B$, then $B^{\perp} \subseteq A^{\perp}$.

(5) If A is a subspace of X, then $A \cap A^{\perp} = \{\theta\}$.

(6) $\left(A^{\perp}\right)^{\perp} = A$.

(7) If x is perpendicular to each $y_1, y_2, \cdots, y_n$, the x is perpendicular to every linear combination of $y_1, y_2, \cdots, y_n$.

Proof. Homework

□

Example 3.1.4. Let $X = \ell_2(\mathbb{C})$, $1 \leq p < \infty$, the space of all complex sequences $a_n = (a_1, a_2, a_3, \cdots)$ for which

$$\sum_{n=1}^{\infty} |a_n|^2 < \infty.$$

Show that the following vectors are orthogonal:

$$e_1 = (\lambda_1, 0, 0, 0, \cdots), e_2 = (0, \lambda_2, 0, 0, \cdots), e_3 = (0, 0, \lambda_3, 0, \cdots), \cdots .$$

Solution: From Example 2.3.5,

$$\langle x, y \rangle = \sum_{n=1}^{\infty} x_n \overline{y_n},$$

for all $x = (x_1, x_2, x_3, \cdots), y = (y_1, y_2, y_3, \cdots) \in \ell_2(\mathbb{C})$.

Thus

$$\langle e_1, e_2 \rangle = (\lambda_1)(0) + (0)(\lambda_2) + 0 + 0 + \cdots = 0$$

$$\langle e_1, e_3 \rangle = (\lambda_1)(0) + (0)(0) + (0)(\lambda_3) + 0 + 0 + \cdots = 0$$

$$\langle e_2, e_3 \rangle = (0)(0) + (\lambda_2)(0) + (0)(\lambda_3) + 0 + 0 + \cdots = 0$$

$$\vdots$$

Therefore, these vectors are orthogonal.

Example 3.1.5. Let $X = \mathbb{R}^3$. Show that the following vectors are orthogonal:

$$x_1 = (1, 2, 2), x_2 = (2, 1, -2), x_3 = (2, -2, 1).$$

Solution: From Example 2.3.3.,

$$\langle x_1, x_2\rangle = (1)(2) + (2)(1) + (2)(-2) = 2 + 2 - 4 = 0.$$

$$\langle x_1, x_3\rangle = (1)(2) + (2)(-2) + (2)(1) = 2 - 4 + 2 = 0.$$

$$\langle x_2, x_3\rangle = (2)(2) + (1)(-2) + (-2)(1) = 4 - 2 - 2 = 0.$$

Hence, x_1, x_2, x_3 are orthogonal.

Lemma 3.1.6. *Let X be a pre-Hilbert space. For two vectors $x, y \in X$, if $x \perp y$, then*

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2,$$

and in general, if $x_1, x_2, \cdots, x_n \in X$ are orthogonal,

$$\left\|\sum_{k=1}^{n} x_k\right\|^2 = \sum_{k=1}^{n} \|x_k\|^2$$

Proof. Suppose that $x, y \in X$ such that $x \perp y$, then

$$\|x + y\|^2 = \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle = \|x\|^2 + \|y\|^2.$$

Assume now, $x_1, x_2, \cdots, x_n \in X$ are orthogonal,

suppose that the equality is true for $n - 1$; that is,

$$\left\|\sum_{k=1}^{n-1} x_k\right\|^2 = \sum_{k=1}^{n-1} \|x_k\|^2$$

set $y = x_n$ and $x = \sum_{k=1}^{n-1} x_k$, then $y \perp x$ (Remark 3.0.3 (7)).

$$\left\|\sum_{k=1}^{n-1} x_k + x_n\right\|^2 = \left\|\sum_{k=1}^{n} x_k\right\|^2 = \|x + y\|^2 = \|x\|^2 + \|y\|^2 = \left\|\sum_{k=1}^{n-1} x_k\right\|^2 + \|x_n\|^2$$

$$\left\|\sum_{k=1}^{n} x_k\right\|^2 = \sum_{k=1}^{n-1} \|x_k\|^2 + \|x_n\|^2 = \sum_{k=1}^{n} \|x_k\|^2$$

Another proof for the second part,

$$\left\|\sum_{k=1}^{n} x_k\right\|^2 = \left\langle \sum_{k=1}^{n} x_k, \sum_{i=1}^{n} x_i \right\rangle$$

using the linearity of the inner product in the first variable,

$$\left\|\sum_{k=1}^{n} x_k\right\|^2 = \left\langle \sum_{k=1}^{n} x_k, \sum_{i=1}^{n} x_i \right\rangle = \sum_{k=1}^{n} \left\langle x_k, \sum_{i=1}^{n} x_i \right\rangle$$

using the linearty of the inner product in the second variable,

$$\left\|\sum_{k=1}^{n} x_k\right\|^2 = \left\langle \sum_{k=1}^{n} x_k, \sum_{i=1}^{n} x_i \right\rangle = \sum_{k=1}^{n} \left\langle x_k, \sum_{i=1}^{n} x_i \right\rangle = \sum_{k=1}^{n} \sum_{i=1}^{n} \langle x_k, x_i \rangle$$

the orthogonality of $x_1, x_2, \cdots, x_n$ implies $\langle x_k, x_i \rangle = 0$ for all $k \neq i$. Thus

$$\left\|\sum_{k=1}^{n} x_k\right\|^2 = \sum_{k=1}^{n} \langle x_k, x_k \rangle = \sum_{k=1}^{n} \|x_k\|^2 .$$

□

Definition 3.1.7. Let X be a pre-Hilbert space and A be a subset of X. Then A is called orthonormal if A is orthogonal (i.e. $\langle x, y \rangle = 0$ for all $x, y \in A$) and $\|x\| = 1$ for all $x \in A$, that is

$$\langle x, y \rangle = \begin{cases} 0 & \text{when } x \neq y, \\ 1 & \text{when } x = y. \end{cases}$$

A sequence finite or infinite of vectors $x_1, x_2, \cdots$ is called an orthonormal sequence if $x_i \perp x_j$ for all $i \neq j$ and $\|x_i\| = 1$ for all i, that is

$$\langle x_i, x_j \rangle = \begin{cases} 0 & \text{when } i \neq j, \\ 1 & \text{when } i = j. \end{cases}$$

More generally, an indexed set, or family, $\{x_\alpha\}, \alpha \in I$ is called orthonormal if $x_\alpha \perp x_\beta$ for all $\alpha, \beta \in I$

with $\alpha \neq \beta$ and $\|x_\alpha\| = 1$ for all $\alpha \in I$, that is

$$\langle x_\alpha, x_\beta \rangle = \delta_{\alpha\beta} = \begin{cases} 0 & \text{when } \alpha \neq \beta, \\ 1 & \text{when } \alpha = \beta. \end{cases}$$

Here, $\delta_{\alpha\beta}$ is the Kronecker delta

Remark 3.1.8. Let X be a pre-Hilbert spaces and $\{a_n\}_{n=1}^{\infty}$ be any orthogonal sequence of non-zero vectors in X. Then the sequence

$$\{b_n\}_{n=1}^{\infty} = \left\{ \frac{a_n}{\|a_n\|} \right\}_{n=1}^{\infty}$$

is orthonormal.

Proof. Homework □

Example 3.1.9. Let $X = \mathbb{R}^3$. Show that the following vectors are orthonormal.

$$x = (\frac{1}{3}, \frac{2}{3}, \frac{2}{3}), \quad y = (\frac{2}{3}, \frac{1}{3}, \frac{-2}{3}), \quad z = (\frac{2}{3}, \frac{-2}{3}, \frac{1}{3}),$$

Solution:

- - - -

Example 3.1.11. Let $C[0, 2\pi]$ be the space of all real valued continuous functions on $C[0, 2\pi]$ with inner product defined by

$$\langle f, g\rangle = \int_0^{2\pi} f(t)g(t)dt$$

Show that the following sequence are orthogonal in $C[0, 2\pi]$,

$$u_n(t) = \cos nt, \quad v_n(t) = \sin nt, \quad n = 0, 1, 2, \cdots$$

and find two orthonormal sequences from u_n, v_n.

Solution: Homework.

Remark 3.1.12.

Recall that a set of vectors is said to be **linearly independent** if no vector in the set can be written as a linear combination of the others. That is, The vectors $v_1, v_2, \cdots, v_n$ in a vector space V are linearly independent if the only solution to the equation $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$ is $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$.

Otherwise, if there exists coefficients (not all zero) satisfying that equation, then the vector are **linearly dependent**.

Example 3.1.13. Let $X = \mathbb{R}^3$. The vectors $e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)$ are linearly independent, since

$$ae_1 + be_2 + ce_3 = (0, 0, 0), \quad \text{implies} \quad a = b = c = 0$$

Example 3.1.14.

Let $X = \mathbb{R}^2$. The vectors $x = (1, 2)$ and $y = (2, 4)$ are linearly dependent, since the coeficients $a = 2, b = -1$ are not both zero, and $2x - y = (0, 0)$.

Lemma 3.1.15.

Let X be a pre-Hilbert space. If $\{e_1, e_2, \cdots, e_n\}$ is an orthonormal set in X, then $\{e_1, e_2, \cdots, e_n\}$ is linearly independent.

Proof. Suppose that

$$\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n = \theta$$

by taking the inner product with a fixed e_m,

$$\left\langle \sum_k \alpha_k e_k, e_m \right\rangle = \langle \theta, e_m\rangle \Longrightarrow \sum_k \alpha_k \langle e_k, e_m\rangle = 0 \Longrightarrow \alpha_m \langle e_m, e_m\rangle = 0,$$

thus $\alpha_m = 0$ for all $m = 1, 2, 3, \cdots, n$. Hence, the set $\{e_1, e_2, \cdots, e_n\}$ is linearly independent. □

Remark 3.1.16.

Let X be a pre-Hilbert space and $\{e_1, e_2, e_3, \cdots\}$ be an orthonormal sequence. For a fixed n, let $x \in \text{span}\{e_1, e_2, \cdots, e_n\}$, so

$$x = \sum_{k=1}^{n} \alpha_k e_k \tag{3.1}$$

by taking the inner product with a fixed e_m,

$$\langle x, e_m \rangle = \left\langle \sum_{k=1}^{n} \alpha_k e_k, e_m \right\rangle = \alpha_m \langle e_m, e_m \rangle = \alpha_m.$$

Thus

$$x = \sum_{k=1}^{n} \langle x, e_k \rangle e_k,$$

this shows that the determination of the coefficients in Equation (3.1) is simple.

Theorem 3.1.17 (Bessel's inequality)**.**

Let X be a pri-Hilbert space and and $\{e_1, e_2, e_3, \cdots\}$ be an orthonormal sequence. Then for every $x \in X$,

$$\sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2 \leq \|x\|^2.$$

Proof. Let $x \in X$, for $n \geq 1$ define the finite "Fourier partial sum"

$$S_n = \sum_{k=1}^{n} \langle x, e_k \rangle e_k.$$

For each $j \leq n$

$$\langle x - S_n, e_j \rangle = \langle x, e_j \rangle - \langle S_n, e_j \rangle = \langle x, e_j \rangle - \left\langle \sum_{k=1}^{n} \langle x, e_k \rangle e_k, e_j \right\rangle,$$

the orthonormality of $\{e_1, e_2, e_3, \cdots\}$ implies

$$\langle x - S_n, e_j \rangle = \langle x - e_j \rangle - \langle x - e_j \rangle = 0,$$

therefore, $x - S_n$ is orthogonal to e_j for all $j \leq n$. Hence, $x - S_n$ is orthogonal to S_n itself.

$$\|x\|^2 = \|(x - S_n) + S_n\|^2 = \|x - S_n\|^2 + \|S_n\|^2 \quad \text{(since } x - S_n \perp S_n\text{)},$$

but $\|x - S_n\|^2 + \|S_n\|^2 \geq \|S_n\|^2$, hence, $\|S_n\|^2 \leq \|x\|^2$.

Now,

$$\|S_n\|^2 = \langle S_n, S_n \rangle = \left\langle \sum_{k=1}^{n} \langle x, e_k \rangle e_k, \sum_{m=1}^{n} \langle x, e_m \rangle e_m \right\rangle = \sum_{k=1}^{n} |\langle x, e_k \rangle|^2 ,$$

hence,

$$\sum_{k=1}^{n} |\langle x, e_k \rangle|^2 \leq \|x\|^2 , \quad \text{for every finite } n$$

Now, it is clear that $\{S_n\}_{n=1}^{\infty}$ is increasing and bounded sequence thus it is convegent and

$$\lim_{n\to\infty} S_n = \sup \{S_1, S_2, \cdots\} \leq \|x\|^2 ,$$

thus

$$\sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2 \leq \|x\|^2 .$$

□

Theorem 3.1.18 (Gram-Schmidt orthonormalization procedure)**.**

Let X be a pre-Hilbert space and $\{x_1, x_2, x_3, \cdots\}$ be a sequence of linearly independent vectors in X. Then there exists an orthonormal sequence $\{e_1, e_2, e_3, \cdots\}$ in x such that span$\{e_1, e_2, e_3, \cdots\}$ = span$\{x_1, x_2, x_3, \cdots\}$. That is $\{e_1, e_2, e_3, \cdots\}$ generat the same linear subspace as $\{x_1, x_2, x_3, \cdots\}$.

Proof. Let $e_1 = \frac{x_1}{\|x_1\|}$, so e_1 is unit vector.

$x_2 = \langle x_2, e_1 \rangle e_1 + u_2 \Longrightarrow u_2 = x_2 - \langle x_2, e_1 \rangle e_1$

u_2 is not zero vector, since $\{x_1, x_2, x_3, \cdots\}$ are linearly independent, also $u_2 \perp e_1$:

$\langle u_2, e_1 \rangle = \langle x_2 - \langle x_2, e_1 \rangle e_1, e_1 \rangle = \langle x_2, e_1 \rangle - \langle \langle x_2, e_1 \rangle e_1, e_1 \rangle = \langle x_2, e_1 \rangle - \langle x_2, e_1 \rangle \langle e_1, e_1 \rangle = 0$

set $e_2 = \frac{u_2}{\|u_2\|}$

Also, $u_3 = x_3 - \langle x_3, e_1 \rangle e_1 - \langle x_3, e_2 \rangle e_2$ and $e_3 = \frac{u_3}{\|u_3\|}$.

In general,

$$u_n = x_n - \sum_{k=1}^{n-1} \langle x_n, e_k \rangle e_k \quad \text{and} \quad e_n = \frac{u_n}{\|u_n\|}$$

□

3.2 The Distance in pre-Hilbert space

Definition 3.2.1. Let X be a normed space and A be a non-empty subset of X. Then the distance from an element $p \in X$ and A is given by

$$\delta = \inf \{d(p, y) : \text{for all y} \in A\} = \inf \{\|p - y\| : \text{for all y} \in A\}$$

Example 3.2.2. Let $X = \mathbb{R}$ and $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \cdots\right\}$. Find the distance between A and $p = 2$.
Solution:

$$\delta = \inf \{d(p, y) : \text{for all y} \in A\} = \inf \{|2 - y| : \text{for all y} \in A\} = \inf \{1, 1.5, 1.6, 1.75, \cdots\}$$

therefore, $\delta = 1$.

Example 3.2.3. Let $X = \mathbb{R}$ and $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \cdots\right\}$. Find the distance between A and $p = 0$.
Solution:

$$\delta = \inf \{d(p, y) : \text{for all y} \in A\} = \inf \{|0 - y| : \text{for all y} \in A\} = \inf \{1, 0.5, 0.333, 0.25, 0.2, \cdots\}$$

therefore, $\delta = 0$.

HOMEWORK

Remark 3.2.4. It is important to know whether there is a $y_0 \in A$ such that $\delta = \|p - y_0\|$, a point which is closet to a given point $p \in X$ and if such element exists, whether it is unique.

Example 3.2.5. Let $X = \{a, b, c\}$ and

$$d(x, y) = \begin{cases} 0 & \text{when } x \neq y, \\ 1 & \text{when } x = y. \end{cases}$$

If $A = \{a, b\}$ and $p = c$, then find the distance between A and p.

$$\delta = \inf \{d(p, y) : \text{for all y} \in A\} = \inf \{1\} = 1.$$

Therefore, $d(c, A) = 1$ and both a, b are at the same minimal distrance, that is the closet point is not unique.

Definition 3.2.6. Let X be a vector space. The line sement joining two given elements a and b in X is defined to be the set of all $x \in X$ of the form:

$$x = \alpha a + (1-\alpha)b, \quad \alpha \in \mathbb{R}, 0 \leq \alpha \leq 1$$

Example 3.2.7. Let $X = \mathbb{R}$. The line segment between $x = 2, b = 5$ is the set of all point x such that

$$\{x = \alpha a + (1-\alpha)b\} = \{2\alpha + 5(1-\alpha), 0 \leq \alpha \leq 1\} = [2, 5]\,.$$

Definition 3.2.8. Let X be a vector space. A subset S of X is said to be convex if for every $a, b \in S$ the line segment joining a and b is contained in S.

Example 3.2.9. Let $X = \mathbb{R}^2$ and $A = \{(x, y) \in \mathbb{R}^2 : y = 2x + 1\}$. Show that A is convex.

Solution:

Let $a = (x_1, y_1), b = (x_2, y_2)$ be any two points in A, thus $y_1 = 2x_1 - 1$ and $y_2 = 2x_2 - 1$.

For $\alpha \in [0, 1]$, the line segment between a and b is the set of all points $z = (x, y) \in \mathbb{R}^2$ such that

$$\{z = \alpha a + (1-\alpha)b\} = \{(x, y) = \alpha(x_1, y_1) + (1-\alpha)(x_2, y_2)\}$$

$$= \{(x, y) = (\alpha x_1 + (1-\alpha)x_2, \alpha y_1 + (1-\alpha)y_2)\}\,.$$

To show the line segment between a and b is in A, we need to show that $z = (x, y) \in A$, that is $y = 2x + 1$.

From the previous equation, $x = \alpha x_1 + (1-\alpha)x_2$ and $y = \alpha y_1 + (1-\alpha)y_2$, since $y_1 = 2x_1 - 1$ and $y_2 = 2x_2 - 1$, then

$$y = \alpha(2x_1 - 1) + (1-\alpha)(2x_2 - 1) = 2\left[(\alpha x_1) + (1-\alpha)x_2\right] - 1 = 2x - 1.$$

Hence $z = (x, y) \in A$.

Example 3.2.10. Let $X = \mathbb{R}^2$ and $S = \{z = (x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$. Show that the unit disk S is convex.

Solution:

$S = \{z \in \mathbb{R}^2 : d(z, 0) \leq 1\} = \{z \in \mathbb{R}^2 : \|z\| \leq 1\}$.

Let $a = (x_1, y_1), b = (x_2, y_2) \in S$, then $d(a, 0) \leq 1$ and $d(b, 0) \leq 1$, that is $\|a\| \leq 1$ and $\|b\| \leq 1$.

For $\alpha \in [0,1]$, the line segment between a and b is the set of all points $z = (x,y) \in \mathbb{R}^2$ such that

$$\{z = \alpha a + (1-\alpha)b\}.$$

To show the line segment between a and b is in S, we need to show that $z \in S$, that is $\|z\| \leq 1$.

$$\|z\| = \|a + (1-\alpha)b\| \leq \|\alpha a\| + \|(1-\alpha)b\| < \alpha(1) + (1-\alpha)(1) \quad (\text{ since } \|a\| \leq 1, \|b\| \leq 1)$$

hence, $\|z\| < 1$, and the line segment between a and b is in S. Therefore, S is convex.

Example 3.2.11. Show that the sequare $C = \{(x,y) : 0 \leq x \leq 1,\ 0 \leq y \leq 1\}$.

Solution: Homework.

Definition 3.2.12. A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be convex if its domain is a convex set and $f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y)$, for all $x, y \in \mathbb{R}^n$ and for all $0 \leq \alpha \leq 1$.

Remark 3.2.13. Let X be a vector space. Then

(1) If Y is a subspace of X, then Y is convex.

Proof: Let $x, y \in Y$ and $\alpha \in [0,1]$.

Then $\alpha x \in Y$ and $(1-\alpha)y \in Y$ (since Y is a subspace),

thus $\alpha x + (1-\alpha)y \in Y$. Hence Y is convex

(2) The intersection of any family (collection) of convex sets is convex.

Proof: Let $\{A_i\}_{i \in I}$ be a collection of convex set. Let

$$C = \bigcap_{i \in I} A_i,$$

we will show that C is convex.

Let $x, y \in C$ and $\alpha \in [a,b]$, then $x, y \in A_i$ for all $i \in I$.

Since each A_i is convex, then $\alpha x + (1-\alpha)y \in A_i$ for all $i \in I$.

This implies that $\alpha x + (1-\alpha)y \in C = \bigcap_{i \in I} A_i$, thus $C = \bigcap_{i \in I} A_i$ is convex.

Remark 3.2.14. The union of convex sets is not necessarily convex. For example, let $A = [2,5], B = [0,1]$. Both A and B are convex subsets of $\mathbb{R}$.

However, $x = 0.5, y = 2.5$ then $x, y \in A \cup B$, but the line segment joining x and y is the interval $[0.5, 2.5]$ which contains points such as 1.5 that do not belong $A \cup B$.

Theorem 3.2.15 (Best Approximation Minimizing Vector)**.**

Let H be a pre-Hilbert space and $M \neq \phi$ be a subset of H which is convex and complete. Then for any $p \in H$, there exists a unique $y_0 \in M$ such that

$$\delta = \inf \{\|p - y\| : y \in M\} = \|p - y_0\|$$

This y_0 is called the best approximation of p from M.

Proof. By the definition of the infimum, there exists a sequence $y_n \in M$ such that $\delta_n = \|p - y_n\| \to \delta$ as $n \to \infty$. We will show that the sequence y_n is Cauchy.

Using parallelogram law:

$$\|(y_m - p) + (p - y_n)\|^2 + \|(y_m - p) - (p - y_n)\|^2 = 2\|y_m - p\|^2 + 2\|p - y_n\|^2$$

$$\|(y_m - p) + (p - y_n)\|^2 = 2\|y_m - p\|^2 + 2\|p - y_n\|^2 - \|(y_m + y_n) - 2p\|^2$$
$$= 2\|y_m - p\|^2 + 2\|p - y_n\|^2 - 4\left\|\frac{1}{2}(y_m + y_n) - p\right\|^2.$$

Since M is convex, then it contains the vector $\frac{1}{2}(y_m + y_n)$, thus $\left\|\frac{1}{2}(y_m + y_n) - p\right\| \geq \delta$ (as δ is infimum).

This implies that

$$\|y_m - y_n\|^2 \leq 2\|y_m - p\|^2 + 2\|p - y_n\|^2 - 4\delta^2,$$

as $n, m \to \infty$, $(y_m - p) \to \delta$ and $(p - y_n) \to \delta$,

hence $2\|y_m - p\|^2 + 2\|p - y_n\|^2 - 4\delta^2 \to 0$, as $n, m \to \infty$.

Therefore, y_n is Cauchy sequence in M and since M is complete, then there exists $y_0 \in M$ such that $y_n \to y_0$ and $(y_n - p) \to (y_0 - p)$, thus $\|y_n - p\| \to \|p - y_0\|$. Since the limit point is unique, then $\delta = \|p - y_0\|$.

For the uniqueness:

Suppose that there exist $z \in M$ such that $z \neq y_0$ and $\|z - p\| = \delta$.

Since M is convex, then $\frac{1}{2}(y_0 + z) \in M$.

Using parallelogram law:

$$\|(y_0 - p) + (p - z)\|^2 + \|(y_0 - p) - (p - z)\|^2 = 2\|y_0 - p\|^2 + 2\|p - z\|^2$$

so

$$\|y_0 - z\|^2 = 2\|y_0 - p\|^2 + 2\|p - z\|^2 - \|(y_0 + z) - 2p\|^2$$

hence

$$\|y_0 - z\|^2 = 2\delta^2 + 2\delta^2 - 4\left\|\frac{1}{2}(y_0 - z) - p\right\|^2$$

Since M is convex, then $\frac{1}{2}(y_0 + z) \in M$, thus

$$\|y_0 - z\|^2 \leq 2\delta^2 + 2\delta^2 - 4\delta^2 = 0.$$

Therefore, $\|y_0 - z\|^2 = 0$ and $y_0 = z$. □

3.3 Direct Sum Decompotions

Definition 3.3.1. Let U and W be two subspaces of a vector space V. Then V is said to be the direct sum of U and W, and written $V = U \oplus W$, if $V = U + W$ and $U \cap W = \{\theta\}$

Lemma 3.3.2. *Let U and W be two subspaces of a vector space V. Then $V = U \oplus W$ if and only if for each vector $v \in V$ there exists a unique vectors $u \in U$ and $w \in W$ such that $v = u + w$.*

Theorem 3.3.3. *Let M be a closed subspace of a Hilbert space H. Then $H = M \oplus M^{\perp}$*

Proof.

Since M is a closed set in a complete space H, then M is complete.

Since M is a subspace, the M is convex.

From the best approximation theorem, for each $x \in H$, there exists a uniqe $y_0 \in M$ such that

$$\|x - y_0\| \leq \|x - y\|, \quad \text{for all } y \in M.$$

Let $z = x - y_0$.

This means that $x = y_0 + z$, where $y_0 \in M$.

We will show that $z \in M^{\perp}$, then we will get $x = y_0 + z$ where $y_0 \in M$ and $z \in M^{\perp}$.

Let y be any vector in M, we will show that $\langle z, y\rangle = 0$.

suppose that λ be a scalar, then:

$$\|z - \lambda y\|^2 = \|z\|^2 - \bar{\lambda}\langle z, y\rangle - \lambda\langle y, z\rangle + |\lambda|^2\|y\|^2$$

we can put $\lambda = \langle z, y\rangle$ and $\|y\| = 1$, then

$$\|z - \lambda y\|^2 = \|z\|^2 - |\langle z, y\rangle|^2 .$$

Now,

$$z - \lambda y = (x - y_0) - \lambda y = x - (y_0 + \lambda y) \ \text{ (from definition of } z)$$

since $y_0 + \lambda y \in M$ (M is a subspace), then

$$\|x - y_0\| \leq \|x - (y_0 + \lambda y)\| \Longrightarrow \|z\| \leq \|z - \lambda y_0\| \Longrightarrow \|z\|^2 \leq \|z - \lambda y_0\|^2$$

but

$$\|z - \lambda y\|^2 = \|z\|^2 - |\langle z, y\rangle|^2 ,$$

then

$$\|z\| \leq \|z\|^2 - |\langle z, y\rangle|^2 ,$$

thus $\langle z, y\rangle = 0$, for all $y \in M$.

Therefore, $z \in M^{\perp}$ and this implies $x = y_0 + z$ where $y_0 \in M$ and $z \in M^{\perp}$.

For the uniqueness, suppose that

$$x = y_1 + z_1 \quad \text{and } x = y_2 + z_2, \quad \text{where } y_1, y_2 \in M \quad \text{and } z_1, z_2 \in M^{\perp}$$

we have, $y_1 + z_1 = y_2 + z_2 \Longrightarrow y_1 - y_2 = z_1 - z_2$

$y_1 - y_2 \in M$ and $z_1 - z_2 \in M^{\perp}$, hence $y_1 - y_2 \in M \cap M^{\perp}$ and $z_1 - z_2 \in M \cap M^{\perp}$.

But $M \cap M^{\perp} = \{\theta\}$, thus $y_1 = y_2$ and $z_1 = z_2$ and $x = y_1 + z_1$ where y_1 and z_1 are unique.

□

Definition 3.3.4. Let M be a closed subspace of a Hilbert space H and let $x \in H$ with $x = y + z$ be the unique decomposition where, $y \in M$ and $z \in M^{\perp}$. Then the vector y is called the orthogonal projection of x in M.

Theorem 3.3.5. *Let M be a closed subspace of a Hilbert space H. For each $x \in H$, Px denote the orthogonal projection of x in M, where $P : H \to H$ is a mapping defined by $P(x) = P(y+z) = y$. This map has the following properties:*

(1) $Py = y$ for all $y \in M$

(2) $Pz = \theta$ *for all* $z \in M^{\perp}$

(3) $P(Px) = Px$

Proof.

(1) Let $y \in M$, the the unique decomposition of y is $y = y + \theta$. Thus $Py = y$ (from definition of P).

(2) Let $z \in M^{\perp}$, the the unique decomposition of z is $z = \theta + z$. Thus $Pz = \theta$ (from definition of P).

(3) For any $x \in H$, $Px \in M$ thus $P(Px) = Px$ (from 1). □

Example 3.3.6. Let $H = R^2$ and Let M be the line $y = x$. Find the orthogonal projection of $x = (2, 1)$ on M

Solution:

$M = \text{ span } \{u\} = \text{ span}\{(1, 1)\}$. Then any point on M look like $\alpha(1, 1) = (\alpha, \alpha), \alpha \in \mathbb{R}^2$.

as $px \in M$, then $Px = \alpha u$, (in this example $u = (1, 1)$).

We know that $z = x - Px = x - \alpha x$ is orthogonal to all vectors in M, thus

$\langle x - \alpha u, u\rangle = 0 \Rightarrow \alpha = \frac{\langle x,u\rangle}{\langle u,u\rangle} \Rightarrow Px = \alpha u = \frac{\langle x,u\rangle}{\langle u,u\rangle} u$

In this example we have $x = (2, 1)$ and $u = (1, 1)$ with the usual inner product, so

$$Px = \frac{\langle (2, 1), (1, 1)\rangle}{\langle (1, 1), (1, 1)\rangle}(1, 1) = (\frac{3}{2}, \frac{3}{2})$$

Example 3.3.7. Let $H = R^2$ and Let M be the line $y = x$. Find the orthogonal projection of $x = (2, 1)$ on M, where $\langle x, y\rangle = 2x_1y_1 + 3x_2y_2$ for all $x, y \in \mathbb{R}^2$

Solution:

$M = \text{ span } \{u\} = \text{ span}\{(1, 1)\}$. Then any point on M look like $\alpha(1, 1) = (\alpha, \alpha), \alpha \in \mathbb{R}^2$.

as $Px \in M$, then $Px = \alpha u$, (in this example $u = (1, 1)$).

We know that

$Px = \frac{\langle x,u\rangle}{\langle u,u\rangle} u$

In this example we have $x = (2, 1)$ and $u = (1, 1)$ with the inner product $\langle x, y\rangle = 2x_1y_1 + 3x_2y_2$, so

$$Px = \frac{\langle (2, 1), (1, 1)\rangle}{\langle (1, 1), (1, 1)\rangle}(1, 1) = (\frac{7}{5}, \frac{7}{5})$$